Local Volatility: A Primer

Here's an introduction to the model that matches option prices across strikes and maturities and lets you price exotic instruments, among other uses. **By BRUNO DUPIRE**

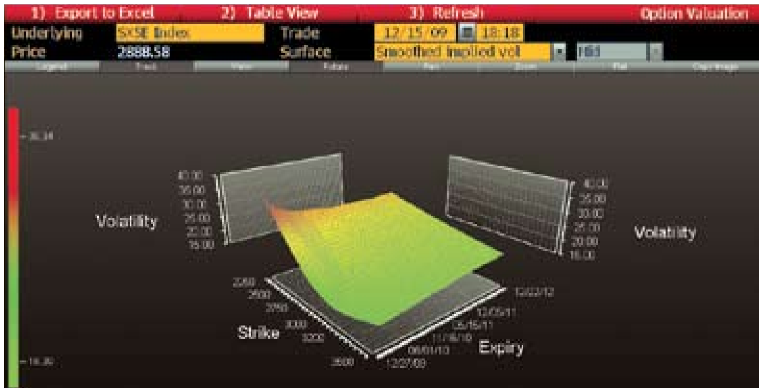
THE LOCAL VOLATILITY model was developed in the early 1990s as a first step toward curing a major weakness of the Black-Scholes options pricing model. The weakness: Black-Scholes is not consistent with market prices of options on the same underlying asset at different strike prices and maturities.

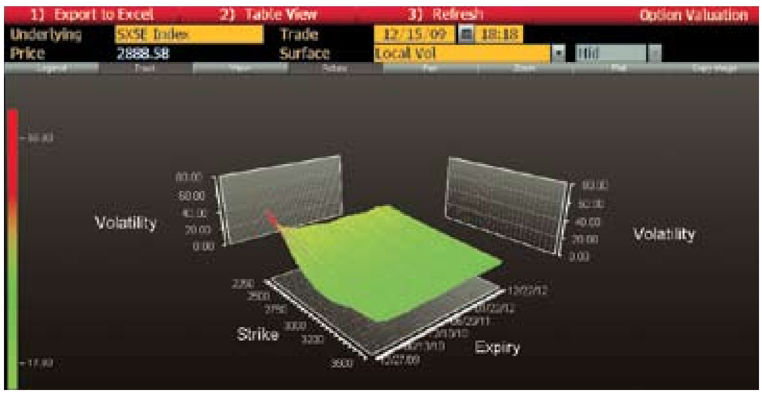
Local volatility models are now implemented in most major banks, and huge portfolios of equity options—contracts that grant the right but not the obligation to buy or sell a stock or index—are valued and risk managed using the framework.

Here's an introduction to the basic concepts of the model and some background on its Bloomberg Professional service implementation, which is available in the Equity and Index Option Valuation (OVME) function. Type SPX <Index> OVME KO <Go>, for example, for a knockout option on the Standard & Poor's 500 Index. A knockout option becomes worthless if the underlying asset price crosses a specified barrier. Click on the arrow to the right of Model and select Local Volatility to use the model to price the option.

Users tend to view the local volatility model in two ways. First, they consider it a model that calibrates automatically to the market: It's the simplest extension of Black-Scholes that's compatible with all vanilla option prices. Second, users view it as a model that provides forward volatilities—or, to be precise, instantaneous variances conditional on a price level—that can be replicated from vanilla options in the same way that instantaneous rates can be replicated from zero coupon bonds.

Instantaneous volatility is a measure of the "vibrations" of the underlying price. It's expressed in annual terms and can be easily converted into typical daily moves. If you divide it by the square root of the number of working days in a year, which is about 16, you obtain the daily standard deviation in percentage terms. For example, a volatility of 20 percent corresponds to about a 1.25 percent daily standard deviation. That means the typical price move over a day is up or down about 1.25 percent.





VOLATILITY SURFACES
Type OVME LVOL <Go> to compare implied volatility, top, with local volatility, bottom, for a selected underlying stock or index.

The Black-Scholes model assumes that the instantaneous volatility is constant. In mathematical terms:

$$\frac{dS}{S} = \mu dt + \sigma dW.$$

THAT ASSUMPTION leads to a major triumph of economics. Fischer Black, Myron Scholes and Robert Merton showed that under the sole assumption of constant volatility the price of an option doesn't depend on the expected rate of return of the underlying asset, μ. It depends on observable market inputs such as interest rates and dividends and relies on only one parameter that must be estimated: volatility, σ.

Given a volatility, we get the option price by applying the Black-Scholes formula. Conversely, the

LOCAL VOL MODEL
Type OVME <Go>, click on the arrow to the right of Model and select Local Volatility to use the model to price an option.

price of an option can be converted into a volatility number by inverting the equation.

The problem is that some options are priced as if there would be an average price change of 1 percent a day, while others are priced as if there would be an average move of 2 percent a day. Because we're talking about a unique price evolution process, though, that's absurd.

The local volatility model stretches the Black-Scholes model by enabling the instantaneous volatility (now called local volatility) to be a function of spot price and of time:

$$\frac{dS}{S} = r(t)\,dt + \sigma(S,t)\,dW.$$

THE FRAMEWORK retains the completeness of Black-Scholes, which gives the ability to perfectly replicate the option by dynamically trading the underlying. That feature attributes a unique value to the option. More important, a judicious choice of the local volatility function ensures fitting the vanilla option prices of all strikes and maturities.

You can use OVME to display the implied volatility surface of the Dow Jones Euro Stoxx 50 Index, for example. Type SX5E <Index> OVME LVOL <Go>, click on the arrow to the right of Surface and select Smoothed Implied Vol. Next, to display the associated local volatility surface, click on the same arrow again and select Local Vol. As you can see from the vertical scale, the local volatility surface is essentially an accentuated version of the implied volatility surface.

Local volatilities are usually considered as the price- and time-dependent input that happens to reprice the vanilla options at their market value. They're actually more than that. They have the status of forward values: in an idealized market (where all strikes and maturities trade with no friction), we can compute and lock unequivocally the local variance (the square of the local volatility) by combining vanilla options of various strikes and maturities. Also, any model that's calibrated to the market (under the sole assumption of no jumps) will be a noisy version of the local volatility model, in the sense that the expectation of the instantaneous variance at a future time for a certain price level is equal to the corresponding local variance. That means that if a user disagrees with the local volatility for a certain date and price level, he can place a trade that reflects that view.

The main uses of the model include pricing exotic options in agreement with vanilla options. You can also use it to lock forward volatilities from observed market prices. You can check whether an implied volatility surface makes sense—whether it's arbitrage-free and smooth. You can analyze the risk of an option position by decomposing the vega across strikes and maturities in so-called super-bucket analysis. (Vega is the change in an option's price for a 1 percent change in volatility.)

Let's look more closely at the pricing of exotic options. Once the model is calibrated to the vanilla options, you can use it to compute more-complicated instruments. In effect, we extrapolate the exotics from the vanilla prices.

Accounting for the market skew can have a large impact on the price of exotics. For example, an up-and-out call option, which is knocked out and becomes worthless if the underlying price moves above a specified barrier, has a positive gamma close to the strike and a negative gamma close to the barrier. A typical equity skew corresponds to high local volatilities close to the strike, which adds value to the option due to the positive gamma and low local volatilities close to the barrier, which is also beneficial to the option holder, as the gamma is negative there. (Gamma is the rate of change in delta given a change in the underlying price. Delta is the change in the option price given a change in the asset price.) The combined effect is that the up-and-out call local volatility price may exceed the price of any Black-Scholes model, whichever volatility input is used.

How does the local volatility model work? The central idea is to extract local volatility from option prices by exploiting both the strike and maturity

Cheat Sheet

Type **STNT <Go>** for a menu of functions and information on structured notes and OTC derivatives.

dimensions. The cost of a calendar spread with two similar maturities depends on the likelihood of getting to the strike at the first maturity and on the volatility that will then prevail.

We obtain the stripping formula, which gives the local volatility as a function of the strike and maturity derivatives of the call prices:

$$\sigma(K,T) = \sqrt{2\frac{\frac{\partial C}{\partial T} + (r-d)K\frac{\partial C}{\partial K} + dC}{K^2\frac{\partial^2 C}{\partial K^2}}}.$$

HOW IS THE MODEL implemented in Bloomberg? Extracting local volatilities from option prices is considered a difficult task—technically, it's an unstable inverse problem. Our approach guarantees the smoothness of the local volatility surface.

We proceed in this way: Build a base by calibrating a stochastic model (Heston) which provides a correct fit and arbitrage-free asymptotes. Compute the residuals (market minus model implied volatilities) for available strikes and maturities and extrapolate them smoothly outside of the strike range. Get a smooth surface of residuals using a nonparametric interpolation. Add the residual surface to the Heston surface to get the implied volatility surface. Strip the implied volatilities to get the local volatility surface. Price the exotic option with these local volatilities.

Market option prices can't be explained by a Black-Scholes model with a single volatility: There's a different Black-Scholes model for each option. In contrast, the local volatility model is the same model for all options because of an instantaneous volatility that depends on the date and the price level. We then use this model to compute more-complicated options, as they can be partially hedged by vanillas. **B**

Bruno Dupire, who developed the local volatility model in the 1990s, is a senior researcher at Bloomberg in New York. bdupire@bloomberg.net